

# Mimicking Friedmann-Robinson-Walker universes with tunable cold Fermi atoms: Galilean invariance fights back

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In a series of papers to demonstrate emergent 'gravity' some authors have explored phonon production in BECs with a tunable speed of sound, in particular to emulate FRW universes. The premiss is that, within such systems, the gapless mode (the phonon) looks Lorentzian for low momentum at least. However, when it comes to phonon production in cold Fermi gases whose speed of sound is controlled by a Feshbach resonance, it is impossible to shake off the underlying Galilean invariance because of the interplay between gapless and gapped modes. Such phonons as are produced do not follow the pattern anticipated for FRW metrics, at variance with the aims of the programme.

## I. INTRODUCTION

In this paper we analyse a Bose condensate as its speed of sound is manipulated experimentally. The reason for so doing is that the low-frequency long-wavelength modes of the phonon field  $\theta$  satisfy the geodesic equation

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} g^{\mu\nu}\partial_\nu\theta) = 0, \quad (1)$$

for some acoustic metric  $g$  determined by the local speed of sound. In consequence, the condensate looks to be an ideal candidate with which to study 'analogue gravity' or 'emergent spacetime', a programme begun more than thirty years ago, when Unruh observed [1] that fluids moving faster than their local speed of sound generated event horizons with many similarities to horizons in general relativity. [A recent review article [2] on this programme has over 700 references.]

The tactics for such an approach with condensates have been addressed by several authors [3–14]. The speed of sound can be varied by changing the density of the condensate [10], but in this paper we examine that class of condensates in which the speed of sound is tuned through the application of an external magnetic field which changes the binding energy of a Feshbach resonance. This allows us to enter a strong coupling regime where the scattering length is larger than the interparticle separation. We are not interested here in event horizons [10] but in the phonon production that arises as we change the metric, mimicking particle production in the early universe. Spatially *homogeneous* condensates with tunable sound speeds and, thereby, simple metrics  $g(t)$ , look prime candidates with which to pursue analogies with FRW universes [12–14].

This paper is, in large part, a commentary on these latter three papers [12–14] which estimate phonon production for such metric behaviour. As discussed in them, some caution is necessary. The equation (1) only describes the linear long-wavelength part of the dispersion

relation. In practice, non-linear dispersion sets in quickly at shorter wavelengths, since it is no more than the reassertion of Galilean invariance of the condensate, temporarily hidden at long wavelengths. Nonetheless, it was anticipated in [13, 14] that there would still remain a narrow regime in which the analogy with FRW universes is sufficiently good to mimic the creation of quantum modes in an inflationary regime.

In order to make a comparison with (1), a robust analytic semi-classical approximation is necessary as a first step. Specifically, the analysis in [12–14] is predicated on the representation of the condensate by a Gross-Pitaevskii (GP) mean bosonic field in which the strength of the self-coupling, and hence the scattering length, is given an explicit time-dependence. In general, for cold bosonic atoms, increasing the scattering length increases the effect of three-body interactions [15]. As a result, even for dilute gases, it is difficult to find circumstances in which the GP mean field approximation is reliable [15] in its unadulterated form. This is not a blanket prohibition. There are situations in which mean-field theory is approximately valid e.g. for bosonic  $^{85}\text{Rb}$  atoms controlled by a Feshbach resonance [16]. However, in general, bosonic quantum evaporation and three-body combination are best accommodated by generalising the GP equation to include non-local terms and terms of higher order respectively [17], which interfere with the simple form of (1). With these qualifications experiments on particle production can be performed whose outcomes can then be predicted (e.g. as in [17]) but they do not sit comfortably in an analogue gravity framework.

This problem at least is largely solved for cold Fermi gases, for which the Pauli exclusion principle strongly suppresses three-body effects [15], permitting robust mean-field approximations [18], and it is these that we shall consider here. Some qualification is necessary, since a semiclassical representation for cold fermionic condensates is, of itself, not enough to lead to simple geodesic equations (1) for the phonon. As we have shown else-

where [19], the hydrodynamic approximation for a condensate of paired cold Fermi atoms is, in general, a two-fluid model which does not permit a representation like (1) with a single metric. A single-fluid model requires a dominant narrow resonance [20], such as the resonance in  ${}^6\text{Li}$  at  $543.25G$ , which we henceforth assume. This gives us the additional benefit that, the narrower the resonance, the more reliable is the mean-field approximation that we shall use [18].

It is also the case that, for cold Fermi gases, the *linear* behaviour of the long-wavelength condensate dispersion relation can also be derived from a GP equation under some circumstances [20]. Although identical to the equation invoked in [12–14] (in terms of the speed of sound), the short-wavelength non-linear behaviour, crucial for understanding the applicability of (1), now has a very different origin. As we shall show, the speed of sound is changed through the coupling of the gapless phonon field to density fluctuations, the collective mode with a finite gap [21]. The deviation from linearity in the dispersion relation is due to these new degrees of freedom in the system, reflected in the presence of a Higgs field [22]. This is very different from the way that the non-linear component of the dispersion relation arises as a consequence of the quantum pressure in conventional Gross-Pitaevskii theory, for which the only degree of freedom is the gapless Goldstone phonon.

The plan of this paper is as follows. In the next sections we show how the geodesic equation (1) can be derived for low energy-momentum phonons in cold Fermi gases controlled by a narrow resonance. In the remainder of the paper we examine to what extent this equation can be justified for modelling phonon production as the speed of sound is varied. Our conclusion is that (1) is deceptive in practice because of the hidden length and time scales, not present in it, that are a consequence of the underlying Galilean invariance of the system.

## II. CONDENSATES FROM COLD FERMION GASES

We begin as in our previous papers [19, 20, 24]. At temperature  $T = 0$  in the narrow resonance limit the cold Fermi gas is described by the action (in units in which  $\hbar = 1$ )

$$\begin{aligned} S = \int dt d^3x \Big\{ & \sum_{\uparrow, \downarrow} \psi_{\sigma}^*(x) \left[ i \partial_t + \frac{\nabla^2}{2m} + \mu \right] \psi_{\sigma}(x) \\ & + \phi^*(x) \left[ i \partial_t + \frac{\nabla^2}{2M} + 2\mu - \nu \right] \phi(x) \\ & - g \left[ \phi^*(x) \psi_{\downarrow}(x) \psi_{\uparrow}(x) + \phi(x) \psi_{\downarrow}^*(x) \psi_{\uparrow}^*(x) \right] \Big\} \quad (2) \end{aligned}$$

The  $\psi_{\sigma}$  denote the Fermi fields with spin label  $\sigma = (\uparrow, \downarrow)$ .

The diatomic field  $\phi$  describes the *narrow* bound-state (Feshbach) resonance with mass  $M = 2m$  and tunable

binding energy  $2\mu - \nu$  which determines the strength of the interactions, and which is controlled by an external magnetic field. Weak fermionic pairing gives a BCS theory of Cooper pairs, and strong fermionic pairing gives a BEC theory of diatomic molecules. The crossover is characterised by the divergence of the *s*-wave scattering length  $a_S \propto (2\mu - \nu)^{-1}$  [18] as it changes sign [25]. On driving the condensate from the deep BCS regime ( $a_S < 0$ ) to the deep BEC regime ( $a_S > 0$ ) by ramping the external magnetic field  $\mathcal{H}$  the speed of sound  $c$  decreases from  $O(v_F)$  to essentially zero.

Integrating out the quadratic Fermi fields gives us an exact non-local one-loop effective action  $S_{NL}$  in terms of  $\phi(x) = -|\phi(x)| e^{i\theta(x)}$  [19, 24]. The dynamics is encoded in  $\theta$ . The action possesses a  $U(1)$  invariance under  $\theta \rightarrow \theta + \text{const.}$ , which is spontaneously broken by spacetime constant *gap* solutions  $|\phi(x)| = |\phi_0| \neq 0$ .

We restrict ourselves to the *mean-field approximation*, the general solution to  $\delta S_{NL} = 0$ , valid for a sufficiently narrow resonance [18, 26]. The Galilean invariants of the theory are the density fluctuation  $\delta|\phi| = |\phi| - |\phi_0|$ ,  $G(\theta) = \dot{\theta} + (\nabla\theta)^2/4m$ , and  $X(\delta|\phi|, \theta) = (\delta|\phi|) + \nabla\theta \cdot \nabla(\delta|\phi|)/2m$ .  $\theta(x)$  is not small. Expanding  $S_{NL}$  in powers of them gives [19, 24] a *local* Galilean invariant effective Lagrangian density  $L_{eff}$  for long-wavelength, low-frequency phenomena of the form

$$\begin{aligned} L_{eff} = & -\frac{1}{2}\rho_0 G(\theta, \epsilon) + \frac{N_0}{4} G^2(\theta, \epsilon) \\ & -\bar{\alpha}\epsilon G(\theta, \epsilon) + \frac{1}{4}\bar{\gamma}X^2(\epsilon, \theta) - \frac{1}{4}\bar{M}^2\epsilon^2, \quad (3) \end{aligned}$$

in which  $\epsilon \propto \delta|\phi|$  i.e. is a density fluctuation. The scaling is chosen so that, on extending  $G(\theta)$  to  $G(\theta, \epsilon) = \dot{\theta} + (\nabla\theta)^2/4m + (\nabla\epsilon)^2/4m$ ,  $\epsilon$  has the same coefficients as  $\theta$  in its spatial derivatives. For what follows, the details of the scaling are immaterial. In (3)  $N_0$  is the density of states at the Fermi surface and  $\rho_0$  is the total fermion number density. The definitions of  $\bar{\alpha}, \bar{\gamma}, \bar{M}$  can be found in [19, 24]. The overbars denote renormalised quantities. Rather than repeat them here we give exemplary plots of the relevant quantities in Fig.1, to be discussed in more detail later.

### A. The hydrodynamic limit

The Euler-Lagrange (EL) equation for  $\theta$  is the continuity equation of a *single* fluid,

$$\frac{\partial}{\partial t} \rho_{\theta} + \nabla \cdot (\rho_{\theta} \mathbf{v}_{\theta}) = 0, \quad (4)$$

where  $\rho_{\theta} = \rho_0 + 2\bar{\alpha}\epsilon - N_0 G(\theta)$  and  $\mathbf{v}_{\theta} = \nabla\theta/2m$ .

The Euler-Lagrange equation for  $\epsilon$  is less transparent, of the form,

$$\frac{d}{dt} [\eta X(\epsilon, \theta)] - \frac{\rho_0}{2m} \nabla^2 \epsilon + \bar{M}^2 \epsilon + 2\bar{\alpha} G(\theta) = 0. \quad (5)$$

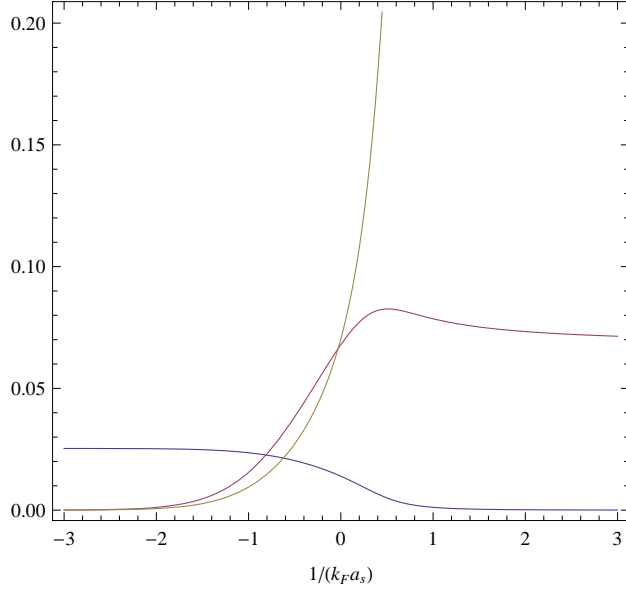


FIG. 1: The curves show  $\bar{\alpha}^2$  (red),  $N_0$  (grey) and  $\bar{\alpha}^2/\bar{M}^2$  (olive) for the value  $\bar{g} = 0.9$  (defined later in terms of  $g$ ) as a function of  $1/k_F a_S$ .

However, if we now neglect both the spatial and temporal variation of  $\epsilon$ , in comparison to  $\epsilon$  itself (the hydrodynamic, or acoustic, approximation) the  $\epsilon$  EL equation becomes

$$\epsilon \approx -2\bar{\alpha}G(\theta)/\bar{M}^2. \quad (6)$$

Inserting this in (5) leads to the Bernoulli equation

$$m\dot{\mathbf{v}}_\theta + \nabla \left[ \delta h + \frac{1}{2} m v_\theta^2 \right] = 0, \quad (7)$$

where the enthalpy is  $\delta h = \delta p/\rho = mc^2 \delta \rho/\rho$ . The resulting equation of state is  $dp/d\rho = mc^2$  across the whole regime.

### B. The acoustic metric

If we now insert (6) into the continuity equation (4) it can be rearranged into the form

$$\frac{d}{dt} \left[ \frac{\rho_0}{c^2} G(\theta) \right] + \nabla \cdot \left[ \frac{\rho_0}{2mc^2} (G(\theta) \nabla \theta) \right] - \nabla \cdot \left[ \rho_0 \nabla \theta \right] = 0 \quad (8)$$

where

$$c^2 = \frac{\rho_0/2m}{N_0 + 4\bar{\alpha}^2/\bar{M}^2}. \quad (9)$$

If we linearise (8) with regard to  $\theta$  in a time-independent condensate it takes the form

$$c^2 \ddot{\theta} - \nabla^2 \theta = 0, \quad (10)$$

with an effective Lorentz metric, enabling us to identify  $c$  as the speed of sound.

More generally, let us suppose that the condensate has a background velocity  $\mathbf{v}_0 = \nabla \theta_0/2m$  in terms of a background phase  $\theta_0$ . If  $\tilde{\theta}$  is the fluctuating phase around the background,

$$\theta = \theta_0 + \tilde{\theta}, \quad (11)$$

then

$$G(\theta) = G(\theta_0) + \dot{\tilde{\theta}} + \mathbf{v}_0 \cdot \nabla \tilde{\theta} = G(\theta_0) + X(\tilde{\theta}, \theta), \quad (12)$$

in terms of  $X$  given earlier.

If we now insert (12) into (8) and linearise with respect to  $\tilde{\theta}$ , (8) then becomes, on using (7),

$$\frac{d}{dt} \left[ \frac{\rho_0}{c^2} X(\tilde{\theta}, \theta) \right] + \nabla \cdot \left[ \frac{\rho_0}{c^2} (\mathbf{v}_0 X(\tilde{\theta}, \theta)) \right] - \nabla \cdot \left[ \rho_0 \nabla \theta \right] = 0 \quad (13)$$

On rearrangement this can be written in the compact covariant form

$$\partial_\mu (f^{\mu\nu} \partial_\nu \tilde{\theta}) = 0, \quad (14)$$

in which

$$\begin{aligned} f^{00} &= -\frac{\rho_0}{c^2}; \quad f^{0i} = -\frac{\rho_0}{c^2} v_0^i; \quad f^{i0} = -\frac{\rho_0}{c^2} v_0^i \\ f^{ij} &= \rho_0 \delta^{ij} - \frac{\rho_0}{c^2} v_0^i v_0^j. \end{aligned} \quad (15)$$

We can rewrite (14) as the geodesic equation

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \theta) = 0, \quad (16)$$

with which we began, where in  $d$  spatial dimensions

$$g_{\mu\nu} = \left( \frac{\rho_0}{c} \right)^{\frac{2}{d-1}} \begin{pmatrix} -(c^2 - v_0^2) & -v_0^j \\ -v_0^i & \delta_{ij} \end{pmatrix} \quad (17)$$

This is the canonical form presented in [2, 12].

### C. Gross-Pitaevskii equations

In what follows we shall restrict ourselves to static homogeneous condensates with constant  $\rho_0$  for which  $\mathbf{v}_0 = \mathbf{0}$ . As we noted elsewhere [24], for such condensates these hydrodynamic results can be derived from a Gross-Pitaevskii (GP) equation on ignoring quantum pressure. Consider the Lagrangian describing the wave-function  $\psi$  of a particle of mass  $2m$ , interacting non-linearly with itself,

$$L(\psi) = i\hbar \psi^* \dot{\psi} - \frac{\hbar^2}{4m} \nabla \psi^* \cdot \nabla \psi - \frac{mc^2}{\rho_0} (|\psi|^2 - \rho_0)^2 \quad (18)$$

where we have restored factors of  $\hbar$ . The Gross-Pitaevskii equation following from (18) is

$$i\hbar \dot{\psi} + \frac{\hbar^2}{2m} \nabla^2 \psi + 2mc^2 \psi - \frac{2mc^2}{\rho_0} \psi |\psi|^2 = 0. \quad (19)$$

If we set  $\psi = \sqrt{\rho} \exp(i\theta)$  and solve (19) at the relevant order in derivatives, we recover (4) and (7). We stress that it is through the slaving of the gapped mode (density fluctuations)  $\epsilon$  to the gapless mode  $\theta$  that we can describe the system (3) by a GP equation with its fewer degrees of freedom.

On introducing a time-dependent phase in  $\psi$  to eliminate the linear term in (19) this GP equation is formally identical with the GP equation proposed for tunable condensates in [12–14], with Lagrangian density

$$L'(\psi) = i\hbar\psi^*\dot{\psi} - \frac{\hbar^2}{4m}\nabla\psi^* \cdot \nabla\psi - \frac{mc^2}{\rho_0}(\psi^*\psi)^2. \quad (20)$$

The immediate difference between the two GP equations lies in the relationship between  $c^2$  and  $a_S$ . For the case of  $L'$ , corresponding to a system of bosonic atoms [12–14]  $c^2$  is *linear* in  $a_S$  as

$$\frac{mc^2}{\rho_0} = \frac{4\pi\hbar^2 a_S}{2m}. \quad (21)$$

In our case of fermionic atoms, the typical behaviour of  $c^2$  as a function of  $a_S$  has the very different behaviour of Fig.2. In Fig.1 we show  $\bar{\alpha}^2, N_0$  and  $\bar{\alpha}^2/\bar{M}^2$  for exemplary values of the condensate parameters, where we have renormalised the coupling strength  $g$  of (3) to  $\bar{g}$  as defined later. In the deep BEC regime, where  $N_0$  is small and  $4\bar{\alpha}^2/\bar{M}^2$  large,  $c(t)$  vanishes. To a good approximation [24],

$$c^2(a_S) = (c_{BCS}^2/2)[1 + \tanh(d(g) - b(g)/k_F a_S)], \quad (22)$$

over the whole range from deep BCS to deep BEC behaviour for a wide spread of couplings  $g$  and it is this parametrisation that we shall use to motivate our results.

#### D. Tuning the condensate

It will be convenient to rewrite  $c^2$  as

$$c^2 = Bc_{BCS}^2. \quad (23)$$

In (23)

$$c_{BCS} = \sqrt{\rho_0/2mN_0^{BCS}} = v_F/\sqrt{3} \quad (24)$$

is the velocity in the deep BCS regime ( $v_F$  is the Fermi velocity) and

$$B = \frac{N_0^{BCS}}{N_0 + 4\bar{\alpha}^2/\bar{M}^2}. \quad (25)$$

$N_0^{BCS}$  is the value of  $N_0$  in the deep BCS regime (see Fig.1).

Applying an external magnetic field  $\mathcal{H}$  changes  $c$  by changing the binding energy  $2\mu - \nu$  which, in turn,

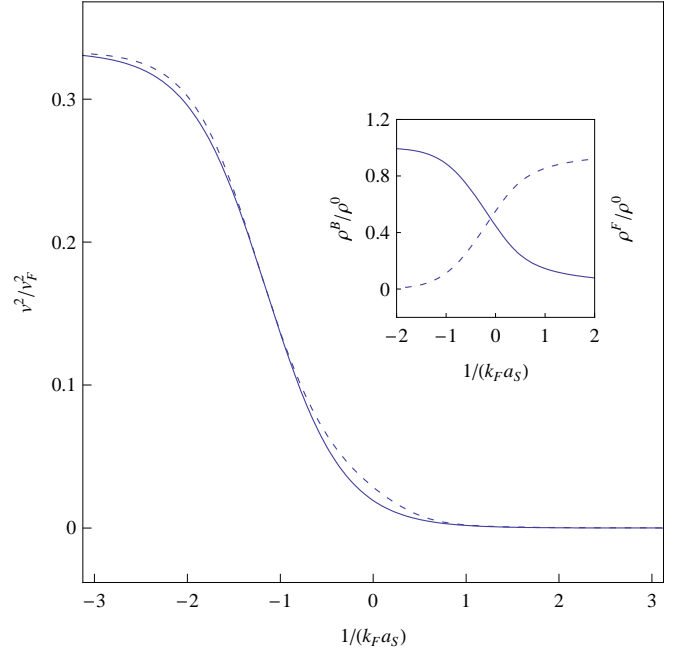


FIG. 2: The dotted line shows  $c^2$  for the value  $\bar{g} = 0.9$  as a function of  $1/k_F a_S$ . The solid line shows the parametrisation (22) for  $d(g) = -1.15$  and  $b(g) = 0.202$ . We get as good or better fits for other values of  $\bar{g}$ , with  $b(g)$  varying by only 25% over the range  $0.2 \leq \bar{g} \leq 1.6$ .

changes the s-wave scattering length  $a_S$  as  $2\mu - \nu = g^2 N_0/k_F a_S$ . In turn, it is determined by the value of the applied external field  $\mathcal{H}$  as [18]

$$a_S = a_{bg} \left( 1 - \frac{\mathcal{H}_\omega}{\mathcal{H} - \mathcal{H}_0} \right), \quad (26)$$

where  $a_{bg}$  is the background (off-resonance) scattering length and  $\mathcal{H}_\omega$  the so-called 'resonance width'.  $\mathcal{H}_0$  is the field required to achieve the unitary limit ( $|a_S| \rightarrow \infty$ ). For the case of interest we pass from the BCS to the BEC regimes as  $\mathcal{H}$  decreases through  $\mathcal{H}_0$ .

For our narrow resonance [18]

$$\frac{1}{k_F a_S} \approx -\frac{1}{k_F a_{bg} \mathcal{H}_\omega} (\mathcal{H} - \mathcal{H}_0), \quad (27)$$

If we adopt (22) the resulting equations can then be solved semi-analytically. We have seen in [20] that, on comparing (22) to the numerical values obtained from (9), it provides behaviour that is better than merely qualitative, even though the *tanh*-behaviour overestimates the rate at which the speed of sound vanishes. This suggests that (22) will overestimate phonon production.

### III. MIMICKING FRW UNIVERSES

Consider a homogeneous condensate in a homogeneous magnetic field  $\mathcal{H}(t)$  varying in time. The phonon field  $\theta$

then satisfies (1) with the metric  $g(t)$  of (17), in which  $c^2(t)$  of (22) is controlled by  $\mathcal{H}$  of (27). To make the situation simpler from the viewpoint of FRW analogue gravity, we follow the authors of [12–14] in assuming that the system is essentially two-dimensional (a pancake condensate). This permits us to use the given speed of sound in a 2D setting with a direct correspondence with the FRW metric. A more general analysis would throw up the same generic results, but with technical complications.

As in [12], taking  $d = 2$  in (17) gives

$$g_{\mu\nu} = \left( \frac{\rho_0}{c_{BCS}} \right)^2 \begin{pmatrix} -c_{BCS}^2 & 0 \\ 0 & a(t)^2 \delta_{ij} \end{pmatrix} \quad (28)$$

If we write

$$c^2(t) = c_{BCS}^2 B(t), \quad (29)$$

then  $a^2(t) = B(t)^{-1}$ .

### A. Constant quench rate - expanding universe

We first consider one of the simplest quenches to implement experimentally in which  $\mathcal{H}$  decreases *uniformly* in time as  $\dot{\mathcal{H}}/\mathcal{H}|_{\mathcal{H}_0} = -\tau_H^{-1}$ . The time dependence of  $B(t)$  as  $\mathcal{H}$  changes is given from (27) as

$$B(t) = (1/2)[1 + \tanh(d(g) - t/\tau_Q)]. \quad (30)$$

$t = 0$  is the time at which the system is at the unitary limit, and

$$\tau_Q = \tau_H \left( \frac{k_F a_{bg} \mathcal{H}_\omega}{b(g) \mathcal{H}_0} \right). \quad (31)$$

The quench parameters are related to the width of the resonance  $\Gamma_0$  by [18]  $\Gamma_0 \approx 4m\mu_B^2 a_{bg}^2 \mathcal{H}_\omega^2 / \hbar^2$ , where  $\mu_B$  is the Bohr magneton. In practice, it is more convenient to work with the dimensionless width  $\gamma_0 \approx \sqrt{\Gamma_0/\epsilon_F}$ . If  $\tau_0 = \hbar/\epsilon_F$ , the inverse Fermi energy (in units of  $\hbar$ ), the relaxation time of the shortest wavelength modes, then

$$\frac{\tau_Q}{\tau_0} = \frac{\tau_Q \epsilon_F}{\hbar} \approx \frac{\pi}{\mu_B \mathcal{H}} \frac{\epsilon_F^2}{\hbar} \gamma_0. \quad (32)$$

To be concrete, consider the narrow resonance in  $^6\text{Li}$  at  $\mathcal{H}_0 = 543.25G$ , discussed in some detail in [27]. [This is to be distinguished from the very broad Feshbach resonance in  $^6\text{Li}$  at  $850G$ .] As our benchmark we take the achievable number density  $\rho_0 \approx 3 \times 10^{12} \text{cm}^{-3}$ , whence  $\epsilon_F \approx 7 \times 10^{-11} \text{eV}$  and  $\gamma_0 \approx 0.2$ . In terms of the dimensionless coupling  $\bar{g}$ , where  $g^2 = (64\epsilon_F^2/3k_F^3)\bar{g}^2$ ,  $^6\text{Li}$  at the density above corresponds to  $\bar{g}^2 \lesssim 1$ . In practice,  $b(g)$  is very insensitive to  $g$ , varying by less than 25% over the range  $\bar{g} = 0.1$  to  $\bar{g} = 0.9$ . For a condensate of density  $\rho$  it follows that  $\bar{g}^2 \propto (\rho_0/\rho)^{1/3}$  and

$$\frac{\tau_Q}{\tau_0} \approx \frac{1}{\mathcal{H}} \left( \frac{\rho}{\rho_0} \right), \quad (33)$$

where  $\dot{\mathcal{H}}$  is measured in units of  $\text{Gauss}(ms)^{-1}$ . Experimentally, it is possible to achieve quench rates as fast as  $\dot{\mathcal{H}} \approx 0.1G/ms$  [27]. As we shall see below, we need such fast quench rates if we are to shake off Galilean invariance. To make calculations simple, we take  $\rho = \rho_0$  in the discussion that follows.

The FRW scale factor  $a(t)$  is, from the above,

$$a^2(t) = B(t)^{-1} = 1 + e^{-2d} e^{2t/\tau_Q}. \quad (34)$$

We have normalised  $a(t)$  to unity in the deep BCS regime. Since  $d \approx -1.15$  then, insofar as the hydrodynamic approximation is reliable, in the BEC regime ( $t > 0$ ) we have

$$a(t) \approx e^{-d} e^{t/\tau_Q}, \quad (35)$$

corresponding to an effective de Sitter universe with Hubble parameter  $H = 1/\tau_Q$ . That is, the simplest experimental situation of a constant quench rate looks to give one of the most interesting analogue models!

The authors of [12–14] also considered instantaneous changes in sound speed, which in our case would formally correspond to replacing the smooth curve of Fig.2 by a step function. Since  $\tau_Q = O(\tau_0)$  is the de facto definition of an instantaneous quench, experimentally we are only an order of magnitude away from effective instantaneous change (and could improve upon this by changing the density).

As an aside we note that, for cold bosonic gases, such as  $^{85}\text{Rb}$  the same parametrisation (26) of scattering length against external field holds (although  $c(t) \propto a_S$ ). Even if it were the case that the GP equation (20) is valid, a stable condensate can only exist when  $a_S > 0$ , i.e. for  $\mathcal{H}_0 < \mathcal{H} < \mathcal{H}_0 + \mathcal{H}_\omega$  in the case of negative  $a_{bg}$  as here. To try to tune  $\mathcal{H}(t)$  to produce a de Sitter-like metric without causing collapse of the condensate must surely be impossible.

However, from the viewpoint of analogue modelling we are not committed to mimicking expanding universes. Contracting or oscillating ‘universes’ are equally acceptable in principle, although we shall see that there are difficulties.

### B. Oscillating fields - cyclic universes

We conclude this section with a brief comment on inducing an oscillation in the speed of sound through an oscillating external field, leading to the metric of a cyclic universe [12]. This is an interesting case in that, with the final and initial states identical, particle production occurs only as a result of parametric excitation.

For reasons that will become clear later, we wish to stay within the BEC regime ( $a_S > 0$ ) without extending too far within it. There is considerable freedom in how we do this, but a simple choice is to take

$$\frac{1}{k_F a_S} = \frac{1}{k_F a_S^0} (1 - \sin \omega t), \quad (36)$$



corresponding from (27) to an oscillating field in which the oscillation of  $a_S^{-1}$  about  $(a_S^0)^{-1}$  extends to the unitary regime.

As a result

$$\begin{aligned} a^2(t) &= 1 + a_0^2 e^{-2A \sin \omega t} \\ &\approx a_0^2 e^{-2A \sin \omega t}, \end{aligned} \quad (37)$$

where  $A = b/k_F a_S$  and  $a_0^2 = \exp -2(d - b/k_F a_S)$ . With  $d = -1.15$  from Fig.2 the approximation is justified.

#### IV. THE BREAKDOWN OF THE SIMPLE ACOUSTIC MODEL

We shall not attempt to perform any calculation of phonon production with respect to the metrics above since, in practice, this simple picture is never implemented. These hydrodynamic equations have arisen from the non-linear EL equations by ignoring derivatives of the density fluctuations. On the other hand, for small fluctuations, the linear approximation (while retaining derivatives of density fluctuations) to the EL equations for  $\theta$  and  $\epsilon$  following from (3) is

$$\begin{aligned} \frac{N_0}{2} \ddot{\theta} - \frac{\rho_0}{4m} \nabla^2 \theta - \bar{\alpha} \dot{\epsilon} &= 0 \\ \frac{\bar{\gamma}}{2} \ddot{\epsilon} - \frac{\rho_0}{4m} \nabla^2 \epsilon + \frac{1}{2} \bar{M}^2 \epsilon + \bar{\alpha} \dot{\theta} &= 0 \end{aligned} \quad (38)$$

where, for simplicity, we have assumed constant coefficients. The residual Galilean invariance lies in the single time derivatives. The identification of the density fluctuations  $\epsilon$  with a gapped mode is clear from (38), although we shall take this no further. A proper discussion of this mode demands that we take the two-fermion cuts into account [20, 22, 23].

On diagonalising (38), we see that for long wavelengths the phonon has dispersion relation  $\omega^2 = c^2 \mathbf{k}^2$  for the *identical* speed of sound  $c$  as in (9), despite the presence of derivatives of  $\epsilon$  that are absent in the hydrodynamic approximation (which corresponds here to  $\epsilon = -2\bar{\alpha}\dot{\theta}/\bar{M}^2$ ). Yet again a more careful analysis shows that this result does not require constant coefficients.

We see that, without having to adopt the hydrodynamic approximation, tuning the speed of sound is implemented microscopically by the coupling of phase (velocity) fluctuations to density fluctuations. Because of this, the intrinsic Galilean invariance introduces time and length scales not visible in the 'Lorentzian' limit of (1), but there nonetheless. We now consider the circumstances in which they cannot be ignored, spoiling the emergence of an FRW geometry. This is a general problem, that has been discussed in part (but then more generally) in [13, 14], but our model allows us to address these issues in a way that is wholly determined from the specific microscopic dynamics of cold Fermi gases.

We first note that phonon creation would be expected to be less important in the BCS and the intermediate

regime, for which  $a(t)$  varies from approximate constancy to approximate linearity in  $t$ , in comparison to the exponential growth of (35). For this and other reasons that will become clear later, in the first instance we restrict ourselves to the BEC regime ( $a_S > 0$ ,  $t > 0$ ) and for which  $N_0 \leq 4\bar{\alpha}^2/\bar{M}^2$  (see Figs. 1). In this regime  $\bar{\alpha}$  is approximately constant. We find that the simple approximations that we shall make are good enough for establishing more than qualitative behaviour. We stress that, unlike the case of cold bosonic atoms, we are not restricting ourselves to  $a_S > 0$  for reasons of stability, since condensates of cold Fermi atoms do not collapse for  $a_S < 0$ .

##### A. Rainbow metrics

The underlying Galilean invariance imposes deviations from a linear phonon dispersion relation, most simply expressed as a variation in  $c$  with wavelength, once we get away from the long wavelength limit. Most simply, we follow [13, 14] in introducing a wavelength dependent metric termed the 'rainbow' metric.

To see the dependence of the speed of sound on wavelength we ignore the second time derivative of  $\epsilon$  in (38) as before, but retain the spatial derivatives, replacing the equations on a mode by mode basis with

$$\begin{aligned} \frac{N_0}{2} \ddot{\theta} - \frac{\rho_0}{4m} \nabla^2 \theta - \bar{\alpha} \dot{\epsilon} &= 0 \\ \frac{\rho_0}{4m} k^2 \epsilon + \frac{1}{2} \bar{M}^2 \epsilon + \bar{\alpha} \dot{\theta} &= 0 \end{aligned} \quad (39)$$

for the mode with wavelength  $k$ .

Eliminating  $\epsilon$  then gives a mode-dependent speed of sound

$$c_k^2 = \frac{\rho_0/2m(1 + \rho_0 k^2/2m\bar{M}^2)}{N_0(1 + \rho_0 k^2/2m\bar{M}^2) + 4\bar{\alpha}^2/\bar{M}^2}. \quad (40)$$

which gives the expansion around  $c$  ( $\equiv c_{k=0}$ ) as

$$c_k^2 = c^2 \left[ 1 + \frac{k^2}{\bar{M}^2} \frac{4\bar{\alpha}^2 c^2}{\bar{M}^2} + \dots \right] \quad (41)$$

In the BEC regime this can be written as

$$c_k^2 \approx c^2 + \left( \frac{\rho_0}{2m} \right)^2 \frac{k^2}{4\bar{\alpha}^2} \quad (42)$$

$$\approx c^2 \left[ 1 + \left( \frac{\rho_0}{2m} \right)^2 \frac{k^2}{4\bar{\alpha}^2 c^2} \right] \quad (43)$$

Away from the unitary limit in the BEC regime,  $\bar{\alpha}$  is approximately constant. After taking the scaling of  $\epsilon$  into account, we find  $\bar{\alpha} \approx \rho_0$ . Provided  $c$  is not too small, when (43) breaks down, this gives us a transitional momentum

$$K \sim \frac{2\bar{\alpha}c}{\rho_0/2m} \approx 4mc \quad (44)$$

above which the short wavelength modes see the non-linear effects of the dispersion relation.

To see the consequences of this in more detail we repeat the analysis of [13, 14]. For modes of wavenumber  $k$  the FRW scale  $a(t)$  is modified in a mode dependent way from (34) to

$$a_k^2(t) \approx [B(t) + k^2/K^2]^{-1} \quad (45)$$

for  $K$  above.

## B. The linear quench

In the BEC regime we find

$$a_k^2(t) \approx \left[ e^{2d} e^{-2t/\tau_Q} + \frac{k^2}{K_0^2} e^{-2d} e^{2t/\tau_Q} \right]^{-1} \quad (46)$$

where

$$K_0 = 4m\bar{a}c_{BCS}/\rho_0 \approx 4mc_{BCS} = (4/\sqrt{3})k_F, \quad (47)$$

with  $k_F$  the Fermi momentum. From our earlier comments, we assume that  $t > 0$ . Some caution is required in that the approximation (46), derived from (41), breaks down when the second term is too large.

Scale factors which may begin as expanding can stall and rapidly contract, as is seen from the 'rainbow' Hubble parameters in the BEC regime,

$$H_k(t) \equiv \frac{\dot{a}_k(t)}{a_k(t)} \approx \frac{-1}{\tau_Q} \frac{\left[ -e^{2d} e^{-2t/\tau_Q} + \frac{k^2}{K_0^2} e^{-2d} e^{2t/\tau_Q} \right]}{\left[ e^{2d} e^{-2t/\tau_Q} + \frac{k^2}{K_0^2} e^{-2d} e^{2t/\tau_Q} \right]}. \quad (48)$$

Taken literally,  $H_k(t)$  decreases monotonically, flipping rapidly from positive to negative values. The transition between an expanding and collapsing 'universe' for wavenumber  $k$  happens at time  $t_k$ ,

$$\frac{t_k}{\tau_Q} = d + \ln \sqrt{\frac{K_0}{k}}, \quad (49)$$

in terms of which

$$H_k(t) \approx \frac{-1}{\tau_Q} \tanh(2(t - t_k)/\tau_Q). \quad (50)$$

From our comments above this approximation breaks down for  $t \gtrsim t_k$ .

There is an infrared bound on  $k$ . For the exemplary condensates with  $N \approx 10^5$  atoms, their width is  $\xi_0 \approx 10^2/k_F$  [20]. This gives

$$0 \lesssim \frac{t_k}{\tau_Q} \lesssim d + \ln 10, \quad (51)$$

or perhaps a little larger. With  $d \approx -1$  this gives

$$0 \lesssim t_k \lesssim \tau_Q, \quad (52)$$

at best (without having to restrict ourselves to the BEC regime *a priori*). Taking the example of Fig.2 with  $b \approx 0.2$  this translates into a transition between an expanding and a collapsing 'universe' for a value of  $t$  for which

$$0 \lesssim 1/a_S k_F \lesssim 5, \quad (53)$$

as we go from the shortest to the longest wavelengths. We stress that  $t_k$  (or the corresponding  $1/a_S k_F$ ) marks the boundary between the applicability of Eq.(1) with its 'Lorentzian' structure and the restoration of Galilean invariance.

As a guide to phonon production during the sweep from BCS to BEC we also need the modified dispersion relation

$$\begin{aligned} \omega_k(t) &\approx \omega_0 \left[ e^{2d} e^{-2t/\tau_Q} + \frac{k^2}{K_0^2} e^{-2d} e^{2t/\tau_Q} \right]^{1/2} \\ &\approx \sqrt{2}\omega_0 e^d e^{-t_k/\tau_Q} \{ \cosh(2(t - t_k)/\tau_Q) \}^{1/2} \end{aligned} \quad (54)$$

where we have adopted the notation of [13, 14], in which  $\omega_0 = |k|c_{BCS}$ . The relevant quantity is the ratio

$$\begin{aligned} \mathcal{R}_k(t) &= \frac{\omega_k(t)}{H_k(t)} \\ &\approx -\sqrt{2} \tau_Q c_{BCS} k \sqrt{\frac{k}{K_0}} \frac{\{ \cosh(2(t - t_k)/\tau_Q) \}^{3/2}}{\sinh(2(t - t_k)/\tau_Q)} \\ &\approx -\frac{2}{\sqrt{3}} \frac{\tau_Q}{\tau_0} \left( \frac{k}{k_F} \right)^{3/2} \frac{\{ \cosh(2(t - t_k)/\tau_Q) \}^{3/2}}{\sinh(2(t - t_k)/\tau_Q)}, \end{aligned} \quad (55)$$

using our definitions of  $t_k$  and  $K_0$  given earlier.

A quantum mode with wavenumber  $k$  only experiences significant amplification (and hence phonon production) when  $\mathcal{R}_k(t) \ll 1$ . As before, this approximation breaks down when  $t \gtrsim t_k$ .

In the vicinity of  $t_k$ , where  $H_k$  is small,  $\mathcal{R}_k$  is corresponding large, as it is for  $t$  much greater than  $t_k$ . In between it achieves a minimum at  $t = t^*$ , where  $\cosh\{2(t^* - t_k)/\tau_Q\} = \sqrt{3}$ . That is,

$$t^* - t_k \approx -\tau_Q/2, \quad (56)$$

at which

$$\mathcal{R}_k(t^*) \approx 3\sqrt{2} \frac{\tau_Q}{\tau_0} \left( \frac{k}{k_F} \right)^{3/2}. \quad (57)$$

[The other minimum at  $t^* - t_k \approx \tau_Q/2$  is unreliable.] In order to have any phonon production we must have as fast a quench as possible, with a current lower bound of  $\tau_Q/\tau_0 \approx 10$  and a lower bound of  $k/k_F$  of  $10^{-2}$ , say, for our typical condensate. Then, for the lowest momentum phonons,

$$\mathcal{R}_k(t^*) \approx 0.04, \quad (58)$$

this minimum increasing as momentum increases. Thus, from (55), there is a window in which  $\mathcal{R}_k(t)$  is sufficiently

small to expect phonon production. However, this window is not large and we note that, if these approximations were reliable for  $t > t_k$  then, for such low momentum phonons we would expect comparable production in both the expanding and contracting phases. There may or may not be a truly contracting phase but, whatever the details, since the contracting phase is purely a consequence of Galilean invariance, it is clear that we are not deriving the particle production that we might have anticipated from Eq.(1). For higher momentum phonons and somewhat slower quenches  $\mathcal{R}_k(t) \gg 1$  throughout and there is no phonon production.

As for reversing the direction of the quench, we have seen that the non-linear effects are greater as  $c$  becomes smaller, making a quench beginning in the deep BEC regime problematical.

### C. Periodic field

For the periodic quench discussed earlier we have

$$a_k^2(t) \approx \left[ a_0^{-2} e^{2A \sin \omega t} + \frac{k^2}{K_0^2} a_0^2 e^{-2A \sin \omega t} \right]^{-1}, \quad (59)$$

from which the rainbow Hubble parameters follow as

$$H_k(t) \approx -A\omega \cos \omega t \frac{\left[ -a_0^{-2} e^{2A \sin \omega t} + \frac{k^2}{K_0^2} a_0^2 e^{-2A \sin \omega t} \right]}{\left[ a_0^{-2} e^{2A \sin \omega t} + \frac{k^2}{K_0^2} a_0^2 e^{-2A \sin \omega t} \right]}, \quad (60)$$

not at all simple, despite its periodicity. We shall not pursue this further here, beyond noting that, from (60), the transitional momentum marking the boundary between Lorentzian and Galilean behaviour is (with  $A \sim 0.1$ )

$$\frac{k}{K_0} \approx e^{2d} \approx 10^{-1}, \quad (61)$$

or somewhat less. With  $10^{-2} \lesssim k/K_0 \lesssim 1$  for our example this means again that the bulk of the activity is controlled by the Galilean group, a conclusion aided by our earlier observation that the parametrisation (22) overestimates particle production.

### D. Time scales

For the linear quench we have already noted that, as we move to the BCS regime and away from exponential behaviour,  $H_k$  diminishes, making phonon production less likely. However, the reason why we have to restrict ourselves to the BEC regime (i.e.  $t \gtrsim 0$ ) concerns the time scales hidden by the Galilean nature of the theory. The coupled equations (38) describe a two-component system of molecules and atom pairs. The density of molecules is  $\rho^B = 2|\phi|^2$ , whose fluctuation is  $\delta\rho^B \propto \epsilon$ . For our spatially *homogeneous* condensate, ignoring damping [23],

these linearised EL equations display the oscillatory behaviour

$$\delta\rho^B = \delta\rho_0^B \cos \Omega t, \quad (62)$$

describing the repeated dissociation of molecules into atom pairs and their reconversion into molecules. The frequency  $\Omega$  of density fluctuations is determined by the energy scale at the beginning of the two-fermion cuts in the energy plane ( $E_{min}$  of [28]). It can be shown [28, 29] that  $\Omega$  increases monotonically from the exponentially damped  $O(\mu \exp(-\pi/2k_F|a_S|))$  in the BCS regime to  $\Omega \approx 2\mu$  in the deep BEC regime  $\Omega \approx 2\mu$ . For the single-fluid model to be valid, the density fluctuations must be able to be averaged to zero on the natural timescale  $\tau = \hbar/Mc^2$  of (18) i.e.  $\tau\Omega \gg 1$ . This is readily achieved in the BEC regime, but is difficult, if not possible, to achieve in the BCS regime [20]. For our exemplary condensate we find  $\tau\Omega \approx 2\pi$  when  $1/k_F a_S \approx 0$ , falling fast as  $1/k_F a_S$  goes negative e.g.  $\tau\Omega \approx 1$  when  $1/k_F a_S \approx -1$  (achieved at time  $t \approx -\tau_Q/5$  for the fastest quenches).

We note that a further reason for our restriction to  $t \gtrsim 0$  is that, in the discussion above, we have taken  $\bar{\alpha} \approx \rho_0$  constant and large. As can be seen from Fig.1,  $\bar{\alpha}$  falls away near the unitarity point. Even had it been sensible to take  $t \lesssim 0$ , reducing  $\bar{\alpha}$  reduces  $K_0$ , effectively making low momentum phonons behave as UV phonons, reducing or eliminating their production in the BCS regime, as we had already anticipated.

In this regard at least, there is no further problem with oscillatory universes in the BEC regime, for which we would require  $\Omega \gg \omega$ . This is automatically the case for the system considered here, since simple calculation shows that  $\tau_0\Omega \gg 1$  for the BEC regime, and we must have  $\omega^{-1} > \tau_0$ . However, this needs further study.

## V. CONCLUSIONS

We find the idea of the emergence of relativistic 'space-time' from condensed matter systems, as manifested in the idealised Eq.(1), beguiling. However, in this paper we have shown that the underlying Galilean invariance of realistic condensed matter systems can impose a brutal reality check.

We have explored this in the context of bosonic condensates of cold Fermi atoms tunable through a narrow Feshbach resonance (a broad resonance requires a doubled metric). To be specific we have considered  $^6\text{Li}$  atoms in concentrations and volumes that are experimentally accessible and with quenches that are equally well achievable with current practice. To have an analytic 'emergence' requires that we have a robust analytic semiclassical approximation and cold Fermi gases permit this in a way that, in general, bosonic atoms do not seem to possess because of many-body effects. In addition, the condensate degrees of freedom for Fermi gases, with a gapped 'Higgs' mode, are very different from those of the



elementary Bose condensates invoked in [12–14], which have provided the counterpoint to this paper. It is only insofar as the gapped mode is slaved to the phonon mode that the Fermi system can be represented by the same Gross-Pitaevskii field as the Bose system, with its limited degrees of freedom. This hides the fact that the speed of sound in condensates of Fermi atoms changes through the coupling of the gapless phonon field to density fluctuations, the collective mode with a finite gap. As a result, cold Fermi and Bose gases have very different mechanisms for acquiring the non-linearities in the phonon dispersion relation, which in the latter case is due to quantum pressure but, in both cases signals a restoration of the underlying Galilean invariance.

From the viewpoint of analogue gravity there is no compulsion to mimic realistic universes; expanding, contracting and oscillatory metrics are equally acceptable. What matters is that we can cast them in the form of (1), make predictions and perform experiments to test those predictions. Unfortunately, once we have taken the additional degrees of freedom of a Fermi system into account we have shown that attempts to mimic particle production in FRW universes by tuning the speed of sound are complicated because of the way in which the underlying Galilean invariance is embedded in the equations of motion.

Our major analysis has been for the simplest experimental field quench, one of constant rate, which induces an acoustic de Sitter metric in the analogue FRW system, one of the better studied systems [12–14]. For low momentum phonons and superfast linear quenches, the best that can happen is that the true effective metric can show a shortlived de Sitter-like expansion before stalling, and perhaps contracting. In fact, the problem is worse since, for sweeps from the deep BCS to deep BEC regions attempting to mimic expanding FRW universes, the hydrodynamic approximation breaks down before reaching the BEC regime, again due to the underlying Galilean invariance. As a result, at most a fraction of the expansion may correspond to the geodesic picture of (1). There are similar problems with oscillating fields and their analogue cyclic universes. There is the caveat that all the above is, in detail, for one  ${}^6\text{Li}$  configuration and it is possible that very different parameter choices (including different Fermi gases) will give better results, despite the generic nature of the problem.

The alternative of looking to tunable cold bosonic gases has its own problems. This is because the inclusion of

many-body effects in a semi-classical description is still not fully resolved. While we should not preclude the possibility of a robust mean-field description in some contexts, the standard way to include these effects takes us out of the simple geodesic description of (1). If these problems can be ignored the results of [12–14] do give a narrow window for success that has evaded us, but more work on the robustness of the approximations needs to be performed.

This should be contrasted to sweeps from the deep BEC towards the BCS regime, for which we have argued [20] that, rather than particle creation, spontaneous vorticity can appear as a consequence of causal horizons. This relies on causality alone, without the need to construct an effective metric.

Finally, we have to say that, despite our attempts to avoid idealisation of condensates and to introduce the complexities which, through Galilean invariance, wreck the analogue gravity programme, we are still dealing with oversimplified systems. Although the early universe may have been homogeneous, condensates are not, requiring traps to contain them. We have avoided discussing traps on the grounds that, if we cannot get a homogeneous system to do what we would like, there is little need to worry about further complications.

We stress that these concerns do not apply to other work by several of the same authors of [12–14], in which they propose physically realisable highly tuned two-component condensates [30, 31] as a way to create massive degrees of freedom, rather than through the Higgs mechanism, as here. This may be a productive way to proceed.

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